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Impedance measurement technique for quantum systems

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Abstract. The impedance measurement technique consists in that the phase-dependent (parametric) inductance of the system is probed by the classical tank circuit via measuring the voltage. The notion of the parametric inductance for the impedance measurement technique is revisited for the case when a quantum system is probed. Measurement of the quantum state of the system of superconducting circuits (qubits) is studied theoretically. It is shown that the result of the measurement is defined by the partial energy levels population in the qubits and by its derivative.

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1 Introduction

The supercurrent I, flowing through a weak link between two bulk superconductors with the phase difference ϕ, has the properties as a nonlinear inductor. This can be described by introducing the phase-dependent (parametric) inductance $\mathcal{L} = (\Phi_0/2\pi)\,(\partial I/\partial\phi)^{-1}$. If the weak link is included in the ring, then the phase ϕ is related to the magnetic flux Φ, piercing the ring, $\phi = 2\pi\Phi/\Phi_0$. The parametric inductance can be measured [1] and being inductively coupled to the resonant LC tank circuit provides the tool to measure the flux Φ [2,3]. The effective inductance of the tank circuit depends on the parametric inductance $\mathcal{L}(\phi)$ and current $I(\phi)$ and thus the measurement in the tank circuit can be used for finding the inductance $\mathcal{L}$ [4], which is the so-called impedance measurement technique.

The impedance measurement technique was recently applied for the measurement of the small currents in mesoscopic samples [5] and was proposed for the description of the currents in superconducting qubits [6–9]; the series of the experimental results were obtained [10]. However the theoretical works in this field consider mostly the ground state. If the superconducting qubit is excited to the upper state, then the current in it has the probabilistic character, and in this way the parametric inductance depends not only on the clockwise (counter-clockwise) current value but also on the probabilities of the respective states. This consideration was used for the description of the phase-biased charge qubit [11]. And in this paper we study in detail the specifics of the impedance measurement technique when the quantum system is probed. The detailed presentation is aimed to show how the tank circuit is influenced by the parametric inductance, and how this inductance have to be treated for the system of coupled superconducting circuits (qubits). For concreteness we consider the superconducting circuits to be either flux or phase-biased charge qubits. The flux qubit [12] consists of a loop with three Josephson junctions. The phase-biased charge qubit [6,7] consists of a loop with two closely situated Josephson junctions and with the gate, which controls the charge on the island between the junctions.

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2 Tank circuit coupled to quantum object

2.1 Equations for tank circuit

The quantum system (coupled superconducting qubits) is considered to be weakly coupled via a mutual inductance M to the classical tank circuit. The circuit consists of the inductor L_T, capacitor C_T, and the resistor R_T connected in parallel. The tank circuit is biased by the current I_{bias}, and the voltage on it V_T can be measured. To obtain the equation for the voltage, we write down the system of equations, for the current in the three branches, namely, through the inductor (I_L), the capacitor (I_C), and the resistor (I_R) (see e.g. in the Chapter 14 of Ref. [2]):

$$I_{bias} = I_L + I_C + I_R, \tag{1}$$

$$I_C = \dot{e},\ e = C_T V_T, \tag{2}$$

$$I_R = V_T/R_T, \tag{3}$$

$$V_T = L_T \dot{I}_L - \dot{\Phi}_e, \tag{4}$$

where e is the charge at the capacitor plate, the dot stands for the time derivative, Φ_e is the flux through the tank circuit. This flux is the response of the quantum system to the flux, induced in it by the current I_L, and its time derivative equals (see below for details):

$$\dot{\Phi}_e = \widetilde{L}\dot{I}_L, \tag{5}$$

and thus equation (4) can be rewritten by introducing the effective inductance of the tank circuit L_{eff}:

$$V_T = L_{eff}\dot{I}_L, \tag{6}$$

$$L_{eff} = L_T - \widetilde{L}. \tag{7}$$

Then from the system of equations (1)–(4) we derive the equation for the voltage in the tank circuit:

$$C_T\ddot{V}_T + R_T^{-1}\dot{V}_T + L_{eff}^{-1}V_T = \dot{I}_{bias}. \tag{8}$$

2.2 Effective inductance of qubits

Now we derive the relation (5); consider the flux $\Phi_e = \sum_i \Phi_e^{(i)}$, where $\Phi_e^{(i)}$ is the flux induced by ith qubit in the tank circuit: $\Phi_e^{(i)} = M_{iT}I_{qb}^{(i)}$. Here M_{iT} is the mutual inductance of the qubit and the circuit, $I_{qb}^{(i)}$ is the current in the ith qubit which equals to the expectation value of the current operator: $I_{qb}^{(i)} = \left\langle \widehat{I}_i \right\rangle = Sp\left(\widehat{\rho}\widehat{I}_i\right)$, where $\widehat{\rho}$ is the reduced density matrix of the system of qubits. (Note that for one qubit, substituting $\Phi_e = MI_{qb}$, equation (8) coincides with Eq. (15) in Ref. [8].)

The total flux that threads the loop of the ith qubit $\Phi^{(i)}$ consists of the external magnetic flux $\Phi_x^{(i)}$ and the self-induced flux $-L_iI_{qb}^{(i)}$ (L_i is the geometrical inductance of the loop):

$$\Phi^{(i)} = \Phi_x^{(i)} - L_iI_{qb}^{(i)}. \tag{9}$$

This equation can be rewritten by introducing the parametric inductance,

$$\mathcal{L}_i^{-1} = \frac{\partial I_{qb}^{(i)}}{\partial \Phi^{(i)}}, \tag{10}$$

to relate the variations of the external flux through the qubit $\delta\Phi_x^{(i)}$ and of the current in it $\delta I_{qb}^{(i)}$, as following:

$$\delta\Phi_x^{(i)} = \delta\Phi^{(i)} + L_i\delta I_{qb}^{(i)} = \left(\mathcal{L}_i + L_i\right)\delta I_{qb}^{(i)}. \tag{11}$$

Thus, we obtain the variation of the flux induced by the qubit in the tank circuit:

$$\delta\Phi_e^{(i)} = M_{iT}\delta I_{qb}^{(i)} = \frac{M_{iT}}{\mathcal{L}_i + L_i}\delta\Phi_x^{(i)}. \tag{12}$$

The flux $\Phi_x^{(i)}$ in the ith qubit is considered to consist of the fluxes induced by the tank circuit, $M_{iT}I_L$, by the microwave source, $\Phi_{ac}^{(i)}\sin\omega t$, and by additional lines and by other qubits, $\Phi_{shift}^{(i)}$ [13]:

$$\Phi_x^{(i)}(t) = M_{iT}I_L(t) + \Phi_{shift}^{(i)} + \Phi_{ac}^{(i)}\sin\omega t. \tag{13}$$

Now let us recall that we consider the variation of the flux in order to calculate the derivative in time $\dot{\Phi}_e$ which have to be substituted in equation (4), which describe the tank circuit. We note that usually the dynamics of the tank circuit (with the frequency ω_{rf} close to the resonant frequency $\omega_T = (L_TC_T)^{-1/2}$) is significantly slower than the dynamics of a qubit driven by the microwave source at frequency $\omega \gg \omega_T$. Hence, being interested in the response of the measurement system (that is of the tank circuit), we average equations over the period $2\pi/\omega$. After this averaging the component $\Phi_{ac}^{(i)}\sin\omega t$ tends to zero and we have: $\delta\Phi_x^{(i)} \approx M_{iT}\delta I_L$. Here and below the time-averaging is assumed. (Note that here it is assumed that the expectation value for the current $I_{qb}^{(i)}$ in equation (12) weakly depend on time during the time interval of the order of $2\pi/\omega$ and its time dependence is defined by the tank circuit dynamics only.) Then it follows

$$\dot{\Phi}_e = \sum_i \frac{\delta\Phi_e^{(i)}}{\delta t} = \sum_i \frac{M_{iT}^2}{\mathcal{L}_i + L_i}\dot{I}_L \tag{14}$$

and equation (5) is obtained with the inductance $\widetilde{L}$, which describes the response of the quantum system to the tank circuit signal, given by:

$$\widetilde{L} = \sum_i \frac{M_{iT}^2}{\mathcal{L}_i + L_i}. \tag{15}$$

Thus, we have obtained the system of equations (7), (8), (10), and (15), which describe the interaction of the classical tank circuit and the quantum circuits (qubits). More accurate (quantum-mechanical) analysis would start the description from the Hamiltonian of the whole system in terms of the operators for both the tank circuit and the qubits with averaging the equations afterwards, as in reference [14] (see also the discussion about similar systems in [15] and [16]). Since this analysis would yield the same equations for the observable values ($V_T = \left\langle \widehat{V}_T \right\rangle$, etc.), we do not consider this procedure here in detail.

3 Analysis of the response of the tank circuit

The measurement consists in biasing the tank circuit with the current $I_{bias} = I_A\cos\omega_{rf}t$ and measuring both the phase shift α and amplitude V_A of the voltage $V_T = V_A\cos(\omega_{rf}t + \alpha)$. (The oscillations can be considered close to the harmonic form due to the small losses in the high-quality tank circuit which is weakly coupled to the nonlinear qubits' inductances, see (19) and (20) below.) Substituting these expressions for I_{bias} and V_T in equation (8) and equating coefficients before $\sin\omega_{rf}t$ and $\cos\omega_{rf}t$, we obtain:

$$\tan\alpha = \frac{R_T}{\omega_{rf}}\left(L_{eff}^{-1} - \frac{\omega_{rf}^2}{\omega_T^2}L_T^{-1}\right), \tag{16}$$

$$V_A = R_TI_A\cos\alpha. \tag{17}$$

Expression for the phase shift is simplified for the tank circuit driven at resonance, $\omega_{rf} = \omega_T = (L_T C_T)^{-1/2}$:

$$\tan\alpha = Q\frac{\widetilde{L}}{L_T - \widetilde{L}} \approx Q\frac{\widetilde{L}}{L_T} = Q\sum_i k_i^2 \frac{L_i \mathcal{L}_i^{-1}}{1 + L_i \mathcal{L}_i^{-1}} \approx Q\sum_i k_i^2 L_i \mathcal{L}_i^{-1}. \quad (18)$$

Here it was assumed that $\widetilde{L} \ll L_T$ and $L_i \ll \mathcal{L}_i$; the latter inequality assumes that qubits' loops have small inductances, the former inequality is justified for large quality factor and small tank circuit-qubit coupling constants:

$$Q = \omega_T R_T C_T \gg 1, \quad (19)$$

$$k_i^2 = \frac{M_{iT}^2}{L_i L_T} \ll 1, \quad (20)$$

and also $k_i^2 Q \lesssim 1$ is assumed.

Alternatively to measuring the phase shift α, equation (18), the qubits' parametric inductances can be probed by measuring the amplitude V_A of the voltage, which is optimal at the frequency $\widetilde{\omega}$ defined by the relation $(\omega_T - \widetilde{\omega})/\omega_T = (2Q)^{-1}$, that is at $\widetilde{\omega} = \omega_T\left(1 - (2Q)^{-1}\right)$, then from equations (16)–(19) it follows:

$$V_A|_{\omega_{rf}=\widetilde{\omega}} \approx R_T I_A \left[1 + \left(1 + Q\frac{\widetilde{L}}{L_T}\right)^2\right]^{-1/2}. \quad (21)$$

This relation can be rewritten, taking into account equation (18) and assuming $\tan\alpha \lesssim 1$ (which is usually the case in experiment [10]) in the form:

$$V_A|_{\omega_{rf}=\widetilde{\omega}} \approx \frac{1}{\sqrt{2}} R_T I_A \left(1 + \frac{1}{2}\tan\alpha|_{\omega_{rf}=\omega_T}\right), \quad (22)$$

which shows the equivalence of the measurements via the amplitude and the phase shift of the tank circuit voltage. Since in practice it is more convenient to probe flux qubits via the phase shift [10,17], we will consider in what follows the phase shift only.

For the case of small geometrical inductances L_i, we can neglect the shielding current, then $\Phi^{(i)} \approx \Phi_x^{(i)}$; we assume the weak coupling of the qubits to the tank circuit (Eq. (20)) and neglect the first term in equation (13); we also define $\Phi_{dc}^{(i)}$ as the constant part of the flux through ith qubit (in practice it changes adiabatically slow); hence $\delta\Phi^{(i)} \approx \delta\Phi_x^{(i)} \approx \delta\Phi_{dc}^{(i)}$ and for calculations equation (18) is supplemented by the relation:

$$\mathcal{L}_i^{-1} \approx \frac{\partial I_{qb}^{(i)}}{\partial \Phi_{dc}^{(i)}}. \quad (23)$$

Note that this value (namely, the r.h.s. of Eq. (23)) can also be interpreted as the magnetic susceptibility [9,15].

4 Inductance of superconducting qubits

Consider the case of a single superconducting qubit in more detail. The phase shift α probes the current in the qubit as following (we rewrite the equations derived above):

$$\tan\alpha \approx k^2 Q L \mathcal{L}^{-1}, \quad (24)$$

$$\mathcal{L}^{-1} \approx \frac{\partial I_{qb}}{\partial \Phi_{dc}}, \quad (25)$$

$$I_{qb} = \left\langle \widehat{I} \right\rangle = Sp\left(\widehat{\rho}\widehat{I}\right). \quad (26)$$

The latter equation can be rewritten for both phase-biased charge qubit [6,7] and flux qubit [12], taking into account that $\widehat{I} = I_{circ}\widehat{\sigma}_z$, as following: $I_{qb} = I_{circ}\langle\widehat{\sigma}_z\rangle$ (here $\widehat{\sigma}_z$ is the Pauli matrix). For a phase-biased charge qubit [11] the circulating current $I_{circ} = I_0$ is phase dependent and equations (24)–(26) show that there are two terms contributing in the tank circuit's phase shift:

$$\tan\alpha \approx k^2 Q L \left(\frac{\partial I_0}{\partial \Phi_{dc}} Z + I_0 \frac{\partial Z}{\partial \Phi_{dc}}\right), \quad (27)$$

where $Z = \langle\widehat{\sigma}_z\rangle$ is the difference between the ground and excited state populations. In a classical system or in the ground state the difference between the energy level's populations is constant, $Z = \text{const.}$, and the second term in equation (27) is zero. In contrast, for the quantum system the interplay of the two terms is essential, which was studied in reference [11]. At this point it is worthwhile to notice that the second term can dominate at resonant excitation, as it was the case in the work [11] (cf. Figs. 3 and 5 in [11]). Hence in some cases this may be the advantage of the impedance measurement technique. Another advantage of the technique may be the possibility of the non-destructive measurement (see in Refs. [9,14,18]).

Consider now the case of a flux qubit in detail. The current operator is defined in the flux basis [12], $\widehat{I} = I_P\widehat{\sigma}_z$, where I_P stands for the amplitude value of the persistent current, and hence the value $\langle\widehat{\sigma}_z\rangle$ defines the difference between the probabilities of the clockwise and counter-clockwise current directions in the loop: $\langle\widehat{\sigma}_z\rangle = P_\downarrow - P_\uparrow = 2P_\downarrow - 1$. Then with equations (24)–(26) we obtain

$$\tan\alpha \approx k^2 Q \frac{L I_P}{\Phi_0} 2\frac{\partial P_\downarrow}{\partial f_{dc}}, \quad (28)$$

where $f_{dc} = \Phi_{dc}/\Phi_0 - 1/2$.

For calculation of the density matrix $\widehat{\rho}$ the Bloch equation is conveniently used (see e.g. Ref. [19]). This equation includes the relaxation and correspondingly is written in the energy basis. Thus, we rewrite equation (28) after introducing the density matrix in the energy representation in terms of the unity matrix $\widehat{1}$ and the Pauli matrices $\widehat{\tau}_i$: $\widehat{\rho} = (1/2)\left(\widehat{1} + X\widehat{\tau}_x + Y\widehat{\tau}_y + Z\widehat{\tau}_z\right)$ (i.e. Z is again the difference between the ground and excited state populations) and obtain:

$$\tan\alpha \approx k^2 Q \frac{L I_P}{\Phi_0}\frac{\partial}{\partial f_{dc}}\left(\frac{2\Delta}{\Delta E}X - \frac{2 I_P \Phi_0 f_{dc}}{\Delta E}Z\right). \quad (29)$$

Here $\Delta E = 2\sqrt{\Delta^2 + (I_P \Phi_0 f_{dc})^2}$ is the distance between the stationary energy levels and Δ is the tunneling amplitude; about the details of this transition from current representation to energy representation see in reference [20].

For one flux qubit in the ground state ($X = 0$, $Z = 1$) it results in the following:

$$\tan\alpha \approx -k^2 QL \frac{\Delta^2 I_P^2}{(\Delta E/2)^3}. \quad (30)$$

It is important to notice that we obtained the result for the ground state, equation (30), which coincides with the earlier obtained results (see Eqs. (3)–(4) in [10]), but in different way – by differentiating the probability $P_\downarrow$, equations (28)–(29). For the description of the flux qubit in the thermal equilibrium one has to put $X = 0$ and $Z = \tanh(\Delta E/2T)$ in equation (29); then by plotting the phase shift versus the magnetic flux, f_{dc}, for different temperatures, one can obtain the suppression and widening of the zero-bias dip (that is in the vicinity of $f_{dc} = 0$) as it was observed in the experiment presented in reference [21] in Figure 3a, which is one more confirmation of our consideration. For example, the zero-bias dip (that is $\tan\alpha$ at $f_{dc} = 0$) is described by the r.h.s. of equation (30) multiplied by the factor $\tanh(\Delta/T)$.

If the first term in the bracket in equation (29) can be neglected (which in concrete case should be checked, but this is usually valid for small driving amplitude Φ_{ac}, see e.g. in [22]), then the expression is simplified:

$$\tan\alpha \approx -k^2 Q \frac{L I_P}{\Phi_0} \left[\frac{\partial}{\partial f_{dc}} \left(\frac{2 I_P \Phi_0 f_{dc}}{\Delta E} \right) Z + \frac{2 I_P \Phi_0 f_{dc}}{\Delta E} \frac{\partial Z}{\partial f_{dc}} \right]. \quad (31)$$

Note that at $f_{dc} = 0$: $\alpha \sim Z$, which means that α probes the changes of the upper level population.

If a qubit is resonantly excited with the driving frequency ω, then the partial energy levels occupation probability Z has the Lorentzian-shape dependence on f_{dc}. It follows that the derivative $\partial Z/\partial f_{dc}$ takes the shape of a hyperbolic-like structure, i.e. it changes from a peak to a dip in the point of the resonance at $\Delta E(f_{dc}) = \hbar\omega$.

5 Conclusion

The impedance measurement technique for the tank circuit being coupled to the system of qubits was studied. The tank circuit was considered to be driven by the rf current and the voltage V_T to be measured. The main results of the work concern the phase shift α and the amplitude V_A of the voltage V_T. It was obtained how the phase shift α is related to the parametric inductances of the qubits $\mathcal{L}_i$, equation (18). It was shown that the dynamics of the qubits can be studied via the amplitude as well as via the phase shift, equation (22). The derivations of these equations were presented in detail in order, first, to make all the assumptions clear (small loops' inductances L_i, weak driving of the tank circuit I_{bias}, high quality factor Q and small couplings k_i, slow dynamics of the tank circuit in comparison with qubits, $\omega_T \ll \omega, \Delta E$), and, second, to show how the parametric inductances of the qubits should be defined, by introducing the difference between the energy levels occupation probabilities, Z. We obtained that the expression for the phase shift α in general contains both terms proportional to Z and proportional to $\partial Z/\partial f_{dc}$, equations (27) and (31). If the latter term dominates, the resonant excitations are visualized as hyperbolic-like structures on the dependence of the phase shift α on the dc flux f_{dc}.

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